Ph.D. Comprehensive Examination in Complex Analysis
Department of Mathematics, Temple University

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Part I: Do three of the following problems

1. Find a harmonic function $u(x, y)$ on the upper half-plane $H = \{x + iy : \ y > 0\}$ such that

$$\lim_{y \to 0^+} u(x, y) = \begin{cases} 0, & \text{if } x > 0 \\ 1, & \text{if } x < 0 \end{cases}.$$

Also find a harmonic conjugate of $u(x, y)$.

2. Let $f(z)$ and $g(z)$ be analytic in an open connected set G. Suppose that for every $z \in G$, $|f(z)| \le |g(z)|$.

(a) Show that there exists a function $h(z)$ analytic in G such that $f(z) = g(z)h(z)$.

(b) Show that either $|f(z)| < |g(z)|$ for all $z \in G$ or there exists a constat c such that $f(z) = cg(z)$.

3. Let z_1, z_2, z_3 be three distinct points on the unit circle $\{z : \ |z| = 1\}$. Recall that for $z \in \mathbb{C}_\infty$ the cross ratio (z, z_1, z_2, z_3) is defined as the image of z under the unique Möbius transformation that takes z_1 to 1, z_2 to 0 and z_3 to ∞.

(a) Determine with proof all values of z such that $(z, z_1, z_2, z_3) \in \mathbb{R}_\infty$.

(b) Suppose $(0, z_1, z_2, z_3) = i$. Find (∞, z_1, z_2, z_3) and the set $\{(z, z_1, z_2, z_3) : \ |z| < 1\}$.

4. Evaluate $\displaystyle\int_0^\infty \frac{\log x}{(1 + x^2)^2}\, dx$.

Part II: Do two of the following problems

1. Show that there does not exist a bijective analytic function from the punctured unit disc $\{z : \ 0 < |z| < 1\}$ to the annulus $\{z : \ 1 < |z| < 2\}$.

2. Let $\{f_n(z)\}$ be a sequence of analytic functions on a region D that converges to a function $f(z)$ uniformly on every compact subset of D. Suppose each $f_n(z)$ has no zeros in D. Prove that either $f(z) \equiv 0$ on D or $f(z)$ has no zeros in D.

3. Give an example of an analytic function $f(z)$ on the open unit disc $D = \{z : \ |z| < 1\}$ such that $f(z) = 0$ if and only if

$$z = 1 - \frac{1}{k} \text{ for } k = 1, 2, 3...$$