PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

August 2002

Part I. Do three (3) of these problems.

I.1. Suppose that $f_n \to f$ a.e. on E and $f_n \to g$ almost uniformly in E.

(1) Give the definition of almost uniform convergence in E.
(2) Prove that $f = g$ a.e. on E.

I.2. Let $f : [0, +\infty) \to \mathbb{R}^+$ be nondecreasing, and such that there exists a positive constant C satisfying

$$\int_{2r}^{4r} f(t)\, dt \le C \int_{r}^{2r} f(t)\, dt,$$

for each $r \ge 0$. Prove that there exists a constant $C' > 0$ such that $f(2r) \le C' f(r)$ for all $r \ge 0$.

I.3. Let $E \subset \mathbb{R}^n$ measurable such that $|E| < \infty$. Prove that $|E \cap B_R(0)^c| \to 0$ as $R \to \infty$; where $B_R(0)^c$ denotes the complement of the Euclidean ball with center 0 and radius R.

I.4. Consider the set $C^{1/2}$ consisting of the functions f's on $[0, 1]$ such that $f(0) = 0$ and

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1/2}} : x \ne y \right\} < \infty.$$

Prove that $(C^{1/2}, \|\cdot\|)$ is complete.

Part II. Do two (2) of these problems.

II.1. Let f_k be measurable and $f_k \to f$ a.e. in $\mathbb{R}^n$. Prove that there exists a sequence of measurable sets $\{E_j\}_{j=1}^{\infty}$ such that $|\mathbb{R}^n \setminus \cup_{j=1}^{\infty} E_j| = 0$ and $f_k \to f$ uniformly on each E_j.

II.2. Let $f : (a, b) \to \mathbb{R}$ be convex and $x \in (a, b)$.

(1) Prove that $\dfrac{f(x+h) - f(x)}{h}$, $h > 0$, decreases with h; and $\dfrac{f(x+h) - f(x)}{h}$, $h < 0$, increases with h.

(2) Prove that the one–sided derivatives

$$D^{\pm} f(x) = \lim_{h \to 0^{\pm}} \frac{f(x+h) - f(x)}{h}$$

exist and satisfy

$$\frac{f(x) - f(x-h)}{h} \leq D^{-} f(x) \leq D^{+} f(x) \leq \frac{f(x+h) - f(x)}{h}, \qquad h > 0.$$

II.3. Let $f \in L^1(0, 1)$ and suppose that $\lim_{x \to 1^-} f(x) = A < \infty$. Prove that

$$\lim_{n \to \infty} n \int_0^1 x^n \, f(x) \, dx = A.$$