Mathematics Real Analysis Ph.D. Qualifying Exam — August, 2006

All functions on $\mathbf{R}^d$ are assumed Lebesgue measurable and all integrals are against Lebesgue measure. You may not use or refer to the Riemann integral in any of your answers; everything must be justified within the context of the Lebesgue theorems (MCT, DCT, LDT, ...).

Part I. (Select 3 questions.)

1. Suppose (f_n) and (g_n) converge uniformly to f and g, respectively, on $[a, b]$.
   (a) Show that if each f_n, $n \geq 1$, is bounded and f is bounded, then (f_n) is uniformly bounded.
   (b) Show that if (f_n) and (g_n) are both uniformly bounded, then $(f_n g_n)$ converges uniformly to fg.

2. Give the construction of the Cantor ternary set C in $[0, 1]$ and prove that
   (a) C is closed.
   (b) C is perfect.
   (c) The Lebesgue measure of C is zero.
   (d) C does not contain any open interval.

3. Let f_n, $n \geq 1$, and f be in $L^1(\mathbf{R}^d)$ and suppose $f_n \to f$ in $L^1(\mathbf{R}^d)$ as $n \to \infty$. Show that there is a subsequence (f_{n_k}) which converges to f almost everywhere.

4. Let f be a continuous function on a compact metric space. Prove that f is uniformly continuous.

Part II. (Select 2 questions.)

1. Let f be a non-negative measurable function defined on $\mathbf{R}^d$. Show that there exist simple measurable functions f_n, $n \geq 1$, defined on $\mathbf{R}^d$, such that f_n increases to f pointwise, as $n \to \infty$.

2. (a) Show that $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$ is finite for $s > 0$.
   (b) Use integration by parts — justifying every step — to show $\Gamma(s+1) = s\Gamma(s)$ for $s > 0$.
   (c) Prove that Γ is continuous on $(0, \infty)$.
   (d) Show that $\Gamma'(s) = \int_0^\infty e^{-x} x^{s-1} \ln x \, dx$ for $s > 0$.

3. Let $i = \sqrt{-1}$ and let
$$F(y) = \int_0^\infty e^{-ty} \frac{1 - e^{it}}{t} \, dt, \qquad y > 0.$$
   (a) Show that $F'(y) = \int_0^\infty e^{-ty}(e^{it} - 1) \, dt$ for $y > 0$.
   (b) Show that
$$F'(y) = \frac{1}{y - i} - \frac{1}{y}, \qquad y > 0.$$
   (c) Let $f(y)$ be the imaginary part of $F(y)$. Show that $f(b) - f(a) = \arctan b - \arctan a$ for $b > a > 0$.
   (d) Show that
$$\int_0^\infty e^{-ty} \frac{\sin t}{t} \, dt = \frac{\pi}{2} - \arctan y, \qquad y > 0.$$