PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

Spring 1994

Part I. Do three (3) of these problems.

I.1. Let

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x < 1. \end{cases}$$

(a) Find $\lim_{n\to\infty} f_n(x)$.
(b) Find $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.
(c) Is there a function $g(x) \in L^1(0,1)$ such that $g(x) \geq f_n(x)$ for all n? Explain briefly.

I.2. Let $f_n(x)$ be a sequence of functions on $\mathbb{R}$ with the propery that for every fixed $x \in \mathbb{R}$ and every subsequence f_{n_k} of f_n, $f_{n_k}(x)$ has a convergent subsequence. Let S be a countable subset of $\mathbb{R}$. Show that there exists a subsequence of f_n that converges at all points of S.

I.3. Show that given δ, with $0 < \delta < 1$, there exists a set $E_\delta \subset [0,1]$, which is perfect [i.e., closed and such that every point of the set is a limit point of the set], nowhere dense and $|E_\delta| = 1 - \delta$.

Hint: The construction is similar to the construction of the Cantor set, except that at the k-th stahe each interval removed has lenght δ times the length of the intervals used in Cantors construction.

I.4. If the *iterated* integrals

$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx \quad \text{and} \quad \int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy$$

exists as finite integrals and are equal to each other, does it necessarily follow that the *multiple* integral

$$\iint f(x,y)\,dx\,dy$$

exists as a finite integral on the square? Cite a theorem, or provide a counterexample to verify a negative response.

Part II. Do two (2) of these problems.

II.1. Let $f \geq 0$ in $\mathbb{R}$, and set

$$g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$$

Show that if $g \in L(\mathbb{R})$ then $f = 0$ a.e.

II.2. For a given function f on $[a, b]$, suppose that there is some $M \geq 0$ such that for all distinct x, y

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M.$$

Explain why

(a) f must be measurable.
(b) f must be differentiable a.e. in $[a, b]$.

II.3. Let f be a continuous function in $[-1, 2]$. Given x, with $0 \leq x \leq 1$, and $n \geq 1$ define the sequence of functions

$$f_n(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\, dt.$$

Show that f_n is continuous in $[0, 1]$ and f_n converges uniformly to f in $[0, 1]$