PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

January 1995

Part I. Do three (3) of these problems.

I.1. Give an example of a closed set which contains no interval and has Lebesgue measure equal to 2.

I.2. State each of the following inequalities. In each instance comment on the case where equality is achieved:

(1) Hölder inequality for L^p, $1 \le p < \infty$
(2) Minkowski's inequality for L^p, $1 \le p < \infty$
(3) Bessel's inequality for L^2

As an alternative to answering *one* of the items above, you may demonstrate that Minkowski's inequality holds for $0 < p < 1$.

I.3. Give an example of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on $[0,1]$ such that f_n converges to some function f in measure but f_n does not converge to f a. e.

I.4. A function $f : \mathbb{R} \to \mathbb{R}$ is measurable if $\{x : f(x) > \alpha\}$ is Lebesgue measurable for each α. Give an example of a measurable function f and a Lebesgue measurable set E such that $f^{-1}(E)$ is nonmeasurable. Hint: Consider the Cantor-Lebesgue function.

Part II. Do two (2) of these problems.

II.1. For each α, θ, $0 \le \alpha \le 1$, $0 \le \theta \le \pi/2$, let $\ell_{\alpha,\theta} : [0,1] \to \mathbb{R}$ be the function illustrated:

Let S be the set of all finite linear combinations $\sum_{k=1}^{n} a_k \ell_{\alpha_k,\theta_k}$ of the $\ell_{\alpha,\theta}$ for all possible choices of a_k, α_k, θ_k. Find the uniform closure of S (*i.e.*, the closure of S with respect to the sup norm). Prove your result.

II.2. Prove: Every nonempty open, bounded, convex set in $\mathbb{R}^2$ that is symmetric about the origin is the open unit ball for some norm on $\mathbb{R}^2$. Hint: use euclidean distance to define $\|x\|$ properly and demonstrate that it is indeed a norm.

II.3. Let f be a nonnegative measurable function defined on $\mathbb{R}$, with $\int_{\mathbb{R}} f < 1$. Let $f_n = f * f * \cdots * f$, convolution n times.

(1) Show that $f_n \to 0$ in $L^1(\mathbb{R})$, as $n \to \infty$

(2) Prove that $f_n \to 0$ a. e., $n \to \infty$.