Real Analysis Ph.D. Qualifying Exam
Temple University
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Part I. (Select 3 questions.)

1. Let v be any nonnegative function defined in all $\mathbf{R}^n$ and for each integer $j > 0$, let B_j denote the ball with center 0 and radius j. Prove that

$$\inf_{x \in B_j} v(x) \to \inf_{x \in \mathbf{R}^n} v(x)$$

as $j \to \infty$.

2. If $\displaystyle\sum_{k=1}^{n} k\, a_k = \frac{n+1}{n+2}$ for $n = 1, 2, \cdots$, then prove that the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

3. Prove that

$$\lim_{n \to \infty} \int_0^{\infty} \frac{\sin(nx)}{1 + n^2 x^3}\, dx = 0.$$

4. Let $f_n(x) = \cos\left(\sqrt{x + 4\,\pi^2\, n^2}\right)$ with $0 \le x < +\infty$. Prove that

   (a) f_n are equicontinuous in $[0, +\infty)$;

   (b) f_n are uniformly bounded;

   (c) $f_n(x) \to 1$ as $n \to \infty$ for each $x \in [0, +\infty)$;

   (d) there is no subsequence of f_n that converges uniformly to 1 in $[0, +\infty)$.
   HINT: if $\sup_{x \in [0,+\infty)} |f_{n_j}(x) - 1| \to 0$ as $j \to \infty$ for a subsequence n_j, then given n_j pick $y_j = \left((2n_j + 1)^2 - 4n_j^2\right)\pi^2$ and notice that $f_{n_j}(y_j) = -1$ contradicting the uniform convergence.

   (e) explain why this does not contradict Arzelá-Ascoli's theorem.

Part II. (Select 2 questions.)

1. Let f_k be a sequence of nonnegative measurable functions in $\mathbf{R}^n$ such that $f_k \to f$ in measure. Prove that $\int_{\mathbf{R}^n} f(x)\,dx \le \liminf_{k\to\infty} \int_{\mathbf{R}^n} f_k(x)\,dx$.

2. Let $f_k \to f$ in $L^1(\mathbf{R}^n)$. Prove that

   (a) $f_k \to f$ in measure;

   (b) $\forall\, \epsilon > 0\ \exists\, t \ge 0$ such that $\int_{\{x:|f_k(x)|\ge t\}} |f_k(x)|\,dx < \epsilon\ \forall\, k$;

   (c) $\forall\, \epsilon > 0\ \exists\, E \subset \mathbf{R}^n$ measurable such that $\int_{\mathbf{R}^n\setminus E} |f_k(x)|\,dx < \epsilon\ \forall\, k$.

3. We say that the sets $A, B \subset \mathbf{R}^n$ are congruent if $A = z + B$ for some $z \in \mathbf{R}^n$.

   Let $E \subset \mathbf{R}^n$ be measurable such that $0 < |E| < +\infty$. Suppose that there exists a sequence of disjoint sets $\{E_i\}_{i=1}^{\infty}$ such that E_i and E_j are congruent for all i, j, and $E = \cup_{j=1}^{\infty} E_j$.

   Prove that all the E_j's are non–measurable.